A family of sequences

Paul Turner
Erindale College, ACT
<paul.turner@ed.act.edu.au>

Perhaps a business colleague threw out a challenge. The year: around 1200. The place: Pisa. The challenge: Calculate how many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on.

The question and its solution found its way into the book *Liber abaci* by Leonardo of Pisa (known as Fibonacci), completed in 1202. It gives rise to the Fibonacci sequence.

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-2} + F_{n-1} & \text{if } n > 2 \end{cases}$$

More recently, a colleague of mine issued the challenge: Prove that the sum of the squares of any two consecutive terms of the Fibonacci sequence is a term of the sequence.

This fact about the sequence would be well known to Fibonacci aficionados, but given here is a more general context in which the sum of consecutive squares property is true, and a surprising connection with Pythagorean triples.

None of the following is at the cutting edge of modern mathematics but the challenge issued by my colleague was within my "zone of proximal development" (Vygotsky, 1978). It led me to investigate with enthusiasm, questions that I would not have thought of otherwise.

On the assumption that students are motivated to learn in much the same way as are mathematics teachers, the not uncommon experience of learning through investigating a challenging question appears to validate the practice of encouraging students to work in a similar research oriented way.

Consider the family of sequences defined recursively by

$$S_{r,n} = \begin{cases} 1 & \text{if } n = 1 \\ r & \text{if } n = 2 \\ S_{r,n-2} + rS_{r,n-1} & \text{if } n > 2 \end{cases}$$

where r is a real number.

The Fibonacci sequence is a member of this family, so what is true of $\{S_{r,n}\}$ is automatically true of $\{F_n\} = \{S_{1,n}\}$.

For any fixed value of r, set $b_n = S_n^2 + S_{n+1}^2$, where S_n and S_{n+1} are consecutive terms of the sequence $\{S_{r,n}\}$. We wish to show that $b_n \in \{S_{r,n}\}$.

We begin by proving the statement:

$$S_{k+1}S_{p-k} + S_k S_{p-(k+1)} = S_p \text{ for } k \in \{1,2,\ldots,p-2\}$$

When $k = 1$, $$S_2 S_{p-1} + S_1 S_{p-2} = rS_{p-1} + S_{p-2} = S_p$$

If it is true that $S_{k+1}S_{p-k} + S_k S_{p-(k+1)} = S_p$ for some arbitrary $k \in \{1,2,\ldots,p-2\}$, then using the definition to replace S_{p-k} we have

$$S_{k+1}(rS_{p-k-1} + S_{p-k-2}) + S_k S_{p-(k+1)} = S_p$$
$$\Rightarrow rS_{k+1}S_{p-(k+1)} + S_{k+1}S_{p-(k+2)} + S_k S_{p-(k+1)} = S_p$$
$$\Rightarrow S_{p-(k+1)}(rS_{k+1} + S_k) + S_{k+1}S_{p-(k+2)} = S_p$$
$$\Rightarrow S_{k+2}S_{p-(k+1)} + S_{k+1}S_{p-(k+2)} = S_p$$

That is, the statement remains true when k is replaced by $k + 1$. Hence, by induction, the statement $S_{k+1}S_{p-k} + S_k S_{p-(k+1)} = S_p$ is true for all $k \in \{1,2,\ldots,p-2\}$.

In particular, when $p = 2n$ and $k = n$, we have $S_{n+1}S_n + S_n S_{n-1} = S_{2n}$.

From this we obtain

$$(rS_n + S_{n-1})S_n + S_n S_{n-1} = S_{2n}$$
$$\Rightarrow rS_n^2 + 2S_{n-1}S_n = S_{2n} \qquad (1)$$

and $$rS_{n+1}^2 + 2S_n S_{n+1} = S_{2n+2} \qquad (2)$$

Adding (1) and (2),

$$r(S_n^2 + S_{n+1}^2) + 2(S_{n-1}S_n + S_n S_{n+1}) = S_{2n} + S_{2n+2}$$
$$\Rightarrow rb_n + 2S_{2n} = S_{2n} + S_{2n+2}$$
$$\Rightarrow rb_n = S_{2n+2} - S_{2n}$$
$$\Rightarrow rb_n = rS_{2n+1} + S_{2n} - S_{2n}$$
$$\Rightarrow b_n = S_{2n+1} \in \{S_{r,n}\}$$

as required.

Another well-known property of the Fibonacci sequence is that consecutive terms are coprime. This is also true in the more general sequence $\{S_{r,n}\}$.

Consider a pair of consecutive terms S_k and S_{k+1} and assume that there is an integer $a > 1$ such that $a \mid S_k$ and $a \mid S_{k+1}$. Then a also divides $S_{k+1} - rS_k = S_{k-1}$. Continuing in this way we must arrive at the conclusion that a divides $S_1 = 1$ which is obviously false. So, the assumption that there exists a divisor of both S_k and S_{k+1} must also be false.

The Pythagoras connection

In an earlier article (Turner, 2006) I showed how the problem of finding isosceles right-angled triangles with integer sides such that the perpendicular sides differ by one, gives rise to the sequence {1, 2, 5, 12, 29, 70, …} with general term $a_n = 2a_{n-1} + a_{n-2}$. This is a member of the family of sequences described above, namely $\{S_{2,n}\}$.

Recall that Pythagorean triples are formed by choosing positive integers m and n, $m > n$, that are coprime with opposite parity, for the triple $(2mn, m^2 - n^2, m^2 + n^2)$. If m and n are consecutive terms from the above sequence, then $2mn$ and $m^2 - n^2$ differ by 1. For example:

m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	5
5	2	21	20	29
12	5	119	120	169
29	12	697	696	985
70	29	4059	4060	5741
169	70	23661	23660	33461

The suspicion arises that the other members of the family of sequences $\{S_{r,n}\}$ might be related to sequences of Pythagorean triples in a similar way.

Suppose integers a and b are the shorter sides of a right-angled triangle. The sides are to be related so that $qa \pm 1 = b$ where q is a positive integer. If we choose to make a an even number then b will be odd for all values of q. Hence, we put $a = 2mn$ and $b = m^2 - n^2$ with the same restrictions on m and n as before, and look for integer solutions for $2qmn \pm 1 = m^2 - n^2$.

Treating this as a quadratic in m, we find the general solutions

$$m = qn + \sqrt{n^2(q^2+1) \pm 1}$$

Observe that $n = 1$, $m = 2q$ is an integer solution to

$$m = qn + \sqrt{n^2(q^2+1) - 1}$$

and that $n = 2q$, $m = 4q^2 + 1$ is an integer solution to

$$m = qn + \sqrt{n^2(q^2+1) + 1}$$

In fact, whenever (n_0, m_0) is an integer solution to $m = qn + \sqrt{n^2(q^2+1) - 1}$, we can put

$$\begin{aligned} m_1 &= qm_0 + \sqrt{{m_0}^2(q^2+1)+1} \\ &= qm_0 + \sqrt{\left(qn_0 + \sqrt{n_0^2(q^2+1)-1}\right)^2(q^2+1)+1} \end{aligned}$$

which, after a great deal of simplification, reduces to

$$\begin{aligned} m_1 &= qm_0 + \sqrt{\left(n_0(q^2+1) + q\sqrt{n_0^2(q^2+1)-1}\right)^2} \\ &= qm_0 + q\left(qn_0 + \sqrt{n_0^2(q^2+1)-1}\right) + n_0 \\ &= 2qm_0 + n_0 \end{aligned}$$

So, one integer solution leads to another.

Similarly, if (n_0, m_0) is an integer solution to $m = qn + \sqrt{n^2(q^2+1)+1}$, we can put

$$m_1 = qm_0 + \sqrt{m_0^{\ 2}(q^2+1)-1}$$

and by the same process arrive at the conclusion $m_1 = 2qm_0 + n_0$, again an integer.

So, for example, in a right-angled triangle if we want side b to differ by 1 from 5 times side a, we would take consecutive terms from the sequence $\{S_{r,n}\}$ with $r = 2 \times 5$ as follows:

$$\{S_{10,n}\}\{1, 10, 101, 1020, 10301, \ldots, 10S_{n-1} + 10S_{n-2}, \ldots\}$$

m	n	$m^2 - n^2$	$2mn$	$5 \times 2mn$
10	1	99	20	100
101	10	10101	2020	10100
1020	101	1030199	206040	1030200
10301	1020	105070201	21014040	105070200
104030	10301	10716130299	2143226060	10716130300

Returning to the relation $qa \pm 1 = b$ above, which says that b will be one away from an integer multiple of a, we may wish instead to explore the possibility of a family of right-angled triangles in which side b is close to a *rational* multiple of a. The reasoning above works equally well if q is rational, provided we are willing to accept triangles with rational rather than integer sides.

So, for example, in a right-angled triangle if we want side b to differ by 1 from $\frac{2}{3}$ of side a, we would take consecutive terms from the sequence

$$\left\{S_{\frac{4}{3},n}\right\} = \left\{1, \frac{4}{3}, \frac{25}{9}, \frac{136}{27}, \frac{769}{81}, \frac{4300}{243}, \ldots\right\}$$

m	n	$m^2 - n^2$	$2mn$	$\frac{2}{3} \times 2mn$
4/3	1	7/9	8/3	16/9
25/9	4/3	481/81	200/27	400/81
136/27	25/9	12871/729	6800/243	13600/729
769/81	136/27	64.76101204…	95.64151806…	63.76101204…

In summary, if q is the ratio of side lengths to be approximated, we can use successive pairs of terms from the sequence $\{S_{r,n}\}$ where $r = 2q$, to substitute for m and n in the triple $(2mn, m^2 - n^2, m^2 + n^2)$.

Finally, a challenge: The sums of squares $b_n = S_n^2 + S_{n+1}^2$ do not in general satisfy the statement $b_{n+1} = b_1 + b_2 + \ldots + b_{n-1} + 2b_n$. Prove that this relation is true for the Fibonacci sequence. Is there a comparable statement that is true for other members of the family $\{S_{r,n}\}$?

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References

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